LECTURE IX

HOMOGENOUS, ISOTROPIC UNIVERSE:

$g_{\mu\nu}$: ~~ROB~~ FRIEDMANN-LEMAÎTRE-ROBERTSON-WALKER METRIC

ENERGY-MOMENTUM TENSOR FOR PERFECT HOMOGENOUS FLUID

$$T_{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

ρ: DENSITY

p: PRESSURE

FROM EINSTEIN'S EQUATION:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{1}{3}\Lambda$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2 R_{c0}^2} + \frac{1}{3}\Lambda$$

FRIEDMANN EQUATIONS

$\Rightarrow$ DYNAMICAL EVOLUTION OF THE UNIVERSE

$a(t) \leftrightarrow \rho(t), p(t)$

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FRIEDMANN EQUATIONS

$\Rightarrow$ DYNAMICAL EVOLUTION OF THE UNIVERSE

$a(t) \leftarrow\!\dashv\ \rho(t), p(t)$

FROM THE TWO EQUATIONS:

$$-\frac{p}{c^2}\frac{da^3}{dt} = \frac{d}{dt}\left(\rho a^3\right)$$

CF.: $-p\,dV = dE$

FLUID WITH <u>NO</u> PRESSURE:

$$\rho(a) = \rho_0\, a^{-3}$$

(DUST, GALAXIES)

PHOTONS: $p = \frac{1}{3}\rho c^2$

$$\rho(a) = \rho_0\, a^{-4}$$

GENERAL EQUATION OF STATE:

$$p = w(a)\,\rho\, c^2$$

$$\Rightarrow \quad \rho = \rho_0 \exp\left[-\int_1^a 3(1+w)\frac{da}{a}\right]$$

DUST: $w=0$

PHOTONS: $w=\frac{1}{3}$

THE REDSHIFT

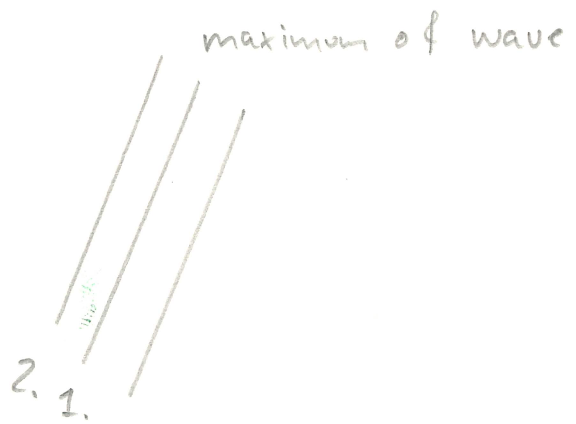
COSMOLOGICAL REDSHIFT z

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e}$$

λ_e : REST FRAME WAVELENGTH OF EMITTED LIGHT

λ_o : OBSERVED WAVELENGTH

RELATION TO SCALE FACTOR



SOURCE : $(r_e, \varphi_e, \theta_e) = \text{const}$

EMITS TWO SIGNALS AT t_e AND $t_e + \tau_e$
(MAXIMUM OF WAVES)

OBSERVED AT $(r_o, \varphi_o, \theta_o) = \text{const}$

AT TIMES : t_o AND $t_o + \tau_o$

PROPERTY OF LIGHT:

MINKOWSKI : $ds^2 = c^2 dt^2 - d\vec{x}^2$

$\Rightarrow \quad ds^2 = 0$ (GENERAL: NULL-GEODESIC)

$$\Rightarrow \quad \frac{c\,dt}{a(t)} = dr$$

CHOOSE: $r_0 = 0$; $d\varphi = d\theta = 0$

$$\Rightarrow \quad \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} = \int_{r_e}^{0} dr = \int_{t_e+\tau_e}^{t_0+\tau_0} \frac{c\,dt}{a(t)}$$

$$\Leftrightarrow \quad \int_{t_e}^{t_e+\tau_e} \frac{c\,dt}{a(t)} + \cancel{\int_{t_e+\tau_e}^{t_0} \frac{c\,dt}{a(t)}} = \cancel{\int_{t_e+\tau_e}^{t_0} \frac{c\,dt}{a(t)}} + \int_{t_0}^{t_0+\tau_0} \frac{c\,dt}{a(t)}$$

SHORT TIME: τ_e, τ_0

$$\Rightarrow \quad \frac{c \cdot \tau_e}{a(t_e)} = \frac{c \cdot \tau_0}{a(t_0)}$$

τ: PERIOD OF LIGHT WAVE: $\lambda = c \cdot \tau$

$$\Rightarrow \quad \frac{a(t_0)}{a(t_e)} = \frac{\lambda_0}{\lambda_e} = 1 + z$$

BASIC COSMOLOGICAL PARAMETERS

(I)

$$H \equiv H(t) = \frac{\dot{a}}{a}$$ HUBBLE PARAMETER

HUBBLE CONSTANT : $H_0 \equiv H(t_0)$

(TODAY)

PLANCK : $67.27 \pm 0.66 \ \frac{km}{s\,Mpc}$

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WMAP9 + BAO : $68.0 \pm 0.7 \ \frac{km}{s\,Mpc}$

CEPHEIDS : $70.6 \pm 3.3 \ \frac{km}{s\,Mpc}$ (NGC 4258)

$73.9 \pm 2.7 \ \frac{km}{s\,Mpc}$ (LMC+MW)

~~FOR~~ FLAT UNIVERSE : $k = 0$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho}{3} + \frac{1}{3}\Lambda$$

$$H^2 = \frac{8\pi G}{3}\left(\rho + \rho_\Lambda\right)$$

WITH : $\rho_\Lambda \equiv \frac{\Lambda}{8\pi G}$

$$\Rightarrow \frac{3H^2}{8\pi G} = \sum_i \rho_i \Rightarrow \rho_{crit} = \frac{3H^2}{8\pi G}$$

$$\rho_{c,0} \equiv \frac{3H_0^2}{8\pi G}$$ Critical DENSITY

$$\boxed{\Omega_{m,0} = \frac{\rho_{m,0}}{\rho_{c,0}}}$$

$\Omega_m = 0.310 \pm 0.008$ (2015)

(PLANCK TT, LowP, BAO)

IN GENERAL:

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \left(\frac{kc^2}{R_c^2}\right) \quad \Big| : H^2$$

$$1 = \frac{8\pi G}{3H^2}\rho + \frac{\Lambda}{3H^2} - \frac{kc^2}{H^2 R_c^2}$$

$$1 = \Omega_m + \Omega_\Lambda + \Omega_k$$

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} \qquad \Omega_k = -\frac{kc^2}{R_c^2 H^2}$$

$\Rightarrow$ $\Omega_m + \Omega_\Lambda < 1 \Rightarrow k < 0$; open UNIVERSE

$\Omega_m + \Omega_\Lambda > 1 \Rightarrow k > 0$; CLOSED UNIVERSE

$\Omega_m + \Omega_\Lambda = 1 \Rightarrow k = 0$ FLAT (EUCLIDEAN) UNIVERSE

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PLANCK + BAO: $\Omega_k = 0.000 \pm 0.005$

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DECELARATION PARAMETER

$$q_0 \equiv - \left(\frac{\ddot{a} a}{\dot{a}^2} \right)_0$$

FROM FRIEDMANN EQN : $p = 0$

$$\ddot{a} = - \frac{4\pi G a}{3} \rho + \frac{1}{3} \Lambda a \quad \Big| \; \frac{a}{\dot{a}^2}$$

$$\Rightarrow \quad q = \frac{\Omega_m}{2} - \Omega_\Lambda \qquad (p = 0)$$

$$\left(= \frac{3}{2} \Omega_m - 1 \quad \text{FLAT} \right)$$

$$q_0 = -0.535$$

(NEGATIVE q_0 $\Rightarrow$ ACCELARATION)

<u>EINSTEIN - de - SITTER UNIVERSE:</u>

$\Omega_m = 1 \; ; \; \Omega_\Lambda = 0$

$$\dot{a}^2 = \frac{8\pi G}{3} \rho a^2$$

(NEGLECT RADIATION) : $\rho = \rho_0 a^{-3}$

$$\Rightarrow \quad a^{1/2} \, da = \left(\frac{8\pi G}{3} \rho_0 \right)^{1/2} dt$$

WITH: $H_0^2 = \frac{8\pi G}{3} \rho_0$

$$\Rightarrow \quad \boxed{a = \left(\frac{3}{2} H_0 t\right)^{2/3}} \qquad (p=0, \Lambda=0, \Omega_m=1)$$

AGE OF THE UNIVERSE:

$$t_0 = \frac{2}{3} \frac{1}{H_0} \qquad (p=0; \Lambda=0; \Omega_m=1)$$

$$H_0 = 50 \frac{km}{s\,Mpc} \rightsquigarrow t_0 \simeq 13\ Gyr$$

$$H_0 = 100 \frac{km}{s\,Mpc} \rightsquigarrow t_0 \simeq 7\ Gyr$$

IF RADIATION DOMINATES: $\rho = \rho_0 a^{-4}$

$$\Rightarrow \quad a = (2H_0 \cdot t)^{1/2} \qquad (\Lambda=0, \Omega_{rad}=1)$$

UNIVERSE DOMINATED BY COSMOLOGICAL CONSTANT

$$\rho_m \simeq 0 \; ; \; \rho_r \simeq 0$$

$$\Rightarrow \quad \dot{a}^2 = \frac{1}{3}\Lambda a^2 - \frac{kc^2}{R_{c,0}^2}$$

$$\Rightarrow \quad a = \frac{kc}{R_{c,0}\sqrt{\Lambda/3}} \sinh\left(\sqrt{\frac{\Lambda}{3}}\, t\right)$$

(FORMALLY OK FOR $\Lambda < 0$ AND $\Lambda > 0$)

$\Rightarrow$ LATE TIMES :

$$a \propto \exp\left(\sqrt{\frac{\Lambda}{3}}\, t\right) \qquad (\Lambda \text{ DOMINATED})$$

$$\frac{\Lambda a^2}{3} \gg \frac{kc^2}{R_{c,0}^2} \qquad \text{AFTER SHORT TIME}$$

$$\Rightarrow \quad \dot{a}^2 = \frac{1}{3}\Lambda a^2$$

(CURVATURE, NEGLIBLE ; SPACE-TIME EUCLIDEAN)

IN GENERAL:

$$a^{3/2}(t) = \sqrt{\frac{\Omega_{m,0}}{1-\Omega_{m,0}}}\ \sinh\left(\frac{3}{2} H_0 \sqrt{1-\Omega_{m,0}}\, t\right)$$

EARLY TIMES:

$$a(t) = \left(\frac{3}{2} H_0 \sqrt{\Omega_{m0}}\right)^{2/3} t^{2/3}$$

LATE TIMES:

$$a(t) = \left(\frac{\Omega_{m0}}{1-\Omega_{m0}}\right)^{1/3} \exp\left(H_0 \sqrt{1-\Omega_{m0}}\, t\right)$$

(VALID AFTER RADIATION)

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