LECTURE VII

NEWTONIAN TREATMENT OF SMALL PERTURBATIONS

~~NO LECTURE DEC, 10TH~~

LARGE SCALE STRUCTURE

SHOW 2dF picture

STUDY SMALL PERTURBATIONS IN FRONT OF SMOOTH BACKGROUND

NON-EXPANDING NEWTONIAN FLUID

EULER EQUATIONS

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} + \frac{1}{\rho} \vec{\nabla} p + \vec{\nabla} \Phi = 0$$

$$\nabla^2 \Phi = 4\pi G \rho$$

ρ: density ; $\vec{u}$: velocity ; p : pressure

Φ : gravitational potential

TRIVIAL SOLUTION: $\vec{u}_0 = 0$, $\rho_0 = \text{const.}$

CHOOSE : $\vec{\nabla} \Phi_0 = 0$

PERTURBATIONS: $\rho = \rho_0 + \delta\rho$, $p = p_0 + \delta p$, $\vec{u} = \vec{u}_0 + \delta\vec{u}$, $\Phi = \Phi_0 + \delta\Phi$

ASSUME: $p = w\rho$ (EQUATION OF STATE)

SOUND SPEED: $c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_{adia}$
(ADIABATIC)

NO SPATIAL VARIATION IN EQUATION OF STATE:

$$c_s^2 = \frac{\delta p}{\delta \rho}$$

INSERT INTO EULER EQN.

($\vec{v}_0, \Phi_0, \rho_0, p_0$ solve eqn to 0th order)

$\Rightarrow$ 1st order:

$$\frac{\partial(\delta\rho)}{\partial t} + \rho_0 \vec{\nabla}\cdot(\delta\vec{u}) = 0$$

$$\frac{\partial(\delta\vec{u})}{\partial t} + \frac{c_s^2}{\rho_0} + \vec{\nabla}(\delta\Phi) = 0$$

$$\nabla^2(\delta\Phi) = 4\pi G\,\delta\rho$$

COMBINE INTO 2nd order DEQN:

$$\ddot{(\delta\rho)} - c_s^2\nabla^2(\delta\rho) = 4\pi G\rho_0\,\delta\rho$$

ANSATZ FOR WAVE EQUATION:

$$\delta\rho(\vec{x},t) = \rho_0\,\delta(\vec{x},t) = A\rho_0 \exp\left[-i\vec{k}\cdot\vec{x} + i\omega t\right]$$

WITH $\delta(\vec{x},t) \equiv \frac{\delta\rho(\vec{x},t)}{\rho_0}$

$\Rightarrow$ DISPERSION RELATION

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0$$

$k = |\vec{k}|$

FOR $\left.\begin{array}{l} \mathrm{Im}(\omega) \neq 0 \\ \mathrm{Re}(\omega) = 0 \end{array}\right\} \Rightarrow$ exponential grows (decays)

$\left.\begin{array}{l} \mathrm{Re}(\omega) \neq 0 \\ \mathrm{Im}(\omega) = 0 \end{array}\right\} \Rightarrow$ OSZILLATION OF SOUND WAVES

ω is imaginary if

$$c_s^2 k^2 - 4\pi G \rho_0 < 0$$

$$\Leftrightarrow k < \boxed{k_J \equiv \sqrt{\frac{4\pi G \rho_0}{c_s^2}}}$$

JEANS WAVE NUMBER

FOR $k \ll k_J$: $\delta\rho$ grows (decays) exponentially

time scale : $\tau_{dyn} = (\mathrm{Im}\,\omega)^{-1/2} \approx \frac{1}{\sqrt{4\pi G \rho_0}}$

JEANS MASS : TOTAL MASS INSIDE SPHERE OF RADIUS : $\lambda_J = \frac{\pi}{k_J}$

$$M_J = \frac{4\pi}{3}\left(\frac{\pi}{k_J}\right)^3 \rho_0 = \frac{\pi^{5/2}}{6} \frac{c_s^3}{G^{3/2} \rho_0^{1/2}}$$

⇒ • PERTURBATIONS WITH $M < M_J$ ARE STABLE AGAINST GRAV. COLLAPSE

• $M > M_J$ COLLAPSE ! TIMESCALE τ_{dyn}

TIMESCALE FOR RESPONSE OF PRESSURE:

PERTURBATION LENGTH λ:

$$\tau_{pressure} \approx \frac{\lambda}{c_s}$$

FOR $\tau_{pressure} > \tau_{dyn}$: NOT ENOUGH TIME TO OPPOSE GRAV. COLLAPSE

EXPANDING NEWTONIAN FLUID

DEFINE: $\rho_0 = \rho_m a^{-3}(t)$

$$\vec{u}_0 = \frac{\dot{a}}{a}\vec{x}$$

$$\nabla\Phi_0 = \frac{4\pi G \rho_0}{3}\vec{x}$$

(only valid for perturbations smaller than size of universe: $|\vec{x}| < H^{-1}$)

$$\Rightarrow \quad \frac{\partial(\delta\rho)}{\partial t} + 3\frac{\dot{a}}{a}\delta\rho + \frac{\dot{a}}{a}(\vec{x}\cdot\vec{\nabla})\delta\rho + \rho_0\vec{\nabla}\cdot(\delta\vec{u}) = 0$$

$$\frac{\partial(\delta\vec{u})}{\partial t} + \frac{\dot{a}}{a}(\delta\vec{u}) + \frac{\dot{a}}{a}(\vec{x}\cdot\vec{\nabla})(\delta\vec{u}) + \frac{c_s^2}{\rho_0}\vec{\nabla}(\delta\rho) + \vec{\nabla}(\delta\Phi) = 0$$

$$\nabla^2(\delta\Phi) = 4\pi G\delta\rho$$

INTRODUCE FOURIER TRANSFORM

IN EXPANDING SPACE:

$$\psi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}} \int \psi_k(t) \exp\left[-\frac{i\vec{k}\vec{x}}{a(t)}\right] d^3r$$

$$r = \frac{k}{a}$$

APPLY TO $\psi = \delta,\ \delta\vec{u},\ \delta\Phi$ with $\delta = \frac{\delta s}{s_0}$

$$\Rightarrow \quad \dot{\delta}_k - \frac{i\vec{k}}{a}\delta\vec{u}_k = 0 \qquad (1)$$

$$\frac{d(a\,\delta\vec{u}_k)}{dt} - i\vec{k}c_s^2\delta_k - i\vec{k}\delta\Phi_k = 0 \qquad (2)$$

$$\delta\phi_k = -\frac{4\pi G s_0}{k^2} a^2 \delta_k \qquad (3)$$

DECOMPOSE PERTURBED VELOCITY INTO ROTATIONAL $\vec{u}_\perp$ AND IRROTATIONAL MODES $\vec{u}_\parallel$

$$\delta\vec{u} = \vec{u}_\perp + \vec{u}_\parallel \quad \text{OR} \quad \delta\vec{u}_k = \vec{u}_\perp(\vec{k}) + \vec{u}_\parallel(\vec{k})$$

PER DEFINITION

$$\vec{\nabla}\cdot\vec{u}_\perp = 0 \qquad \vec{\nabla}\times\vec{u}_\parallel = 0$$

OR FT:

$$\vec{k}\cdot\vec{u}_\perp(\vec{k}) = 0 \qquad \vec{k}\cdot\vec{u}_\parallel(\vec{k}) = |\vec{k}|\,|\vec{u}_\parallel|$$

$\perp$ ✓ $\parallel$ ✓

INTO EQN (1):

$$\frac{i\vec{k}}{a}\left(\vec{u}_{\parallel}(\vec{k})+\vec{u}_{\perp}(\vec{k})\right)=\dot{\delta}_k$$

$$\Leftrightarrow \frac{i\vec{k}}{a}\vec{u}_{\parallel}(\vec{k})=\dot{\delta}_k$$

$$\Leftrightarrow \frac{ik}{a}|\vec{u}_{\parallel}(k)|=\dot{\delta}_k \qquad (*)$$

~~$\vec{u}_{\perp}$ decouples~~ FROM (2): (SEPERATE IN MODE $\parallel$ AND $\perp$ TO $\vec{k}$!

$$\Rightarrow \frac{d\left(a\,\vec{u}_{\perp}(\vec{k})\right)}{dt}=0 \Rightarrow \vec{u}_{\perp}(\vec{k}) \propto a^{-1}(t)$$

$$\Rightarrow \delta\vec{u}_k=\frac{a\,\dot{\delta}_k}{ik}\frac{\vec{k}}{k}+\frac{\overrightarrow{\text{const.}}}{a(t)} \qquad \overrightarrow{\text{const}} \perp \vec{k}$$

THE ROTATIONAL MODE IS DECOUPLED AND DECAYS LIKE a^{-1} !

ONLY USE IRROTATIONAL MODE AND DROP LABEL $\parallel$

FURTHER ONLY WORK IN FOURIER SPACE AND DROP INDEX "k"

$$\Rightarrow \ddot{\delta}_k+2\frac{\dot{a}}{a}\dot{\delta}_k+\left(\frac{c_s^2k^2}{a^2}-4\pi G\bar{\rho}_0\right)\delta_k=0$$

↑ acts like friction term!

RECOVER PREVIOUS RESULT WITH $\dot{a}=0$

AND IDENTIFY $\frac{|\vec{k}|}{a}$ WITH PHYSICAL WAVENUMBER : $\vec{k}$: COMOVING WAVENUMBER

* JEANS WAVENUMBER:

$$k_J^2 \equiv \frac{4\pi G \rho_0 a^2}{c_s^2}$$

AS BEFORE SEPARATES GRAV. STABLE AND UNSTABLE MODES.

IF $a \sim t^n$ FOR $k \gg k_J$ THE SOUND WAVES OSCILLATE WITH

$$\omega \approx \frac{c_s k}{a(1-n)}$$

(EXACT SOLUTION BESSEL FCT. SEE LATER)

FOR $k \ll k_J$ UNSTABLE GROWING MODES:

EXAMPLE: FLAT, MATTER DOMINATED UNIVERSE:

$$\frac{\dot{a}}{a} = \left(\frac{2}{3t}\right) \; ; \quad \rho_0 = \frac{1}{6\pi G t^2}$$

$$\Rightarrow \quad \ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2} = 0$$

Use $k \ll k_J$ or $\frac{c_s^2 k^2}{a^2} \ll 4\pi G \rho_0$

TWO SOLUTIONS:

GROWING MODE: $\delta_+(t) = \delta_+(t_i)\left(\frac{t}{t_i}\right)^{2/3}$

DECAYING MODE: $\delta_-(t) = \delta_-(t_i)\left(\frac{t}{t_i}\right)^{-1}$ ← NOT IMPORTANT FOR LONG TERM BEHAVIOUR

⇒ EXPANSION OF UNIVERSE SLOWS exponential GROWTH TO POWER-LAW GROWTH!

MULTIPLE COMPONENTS

$$\ddot{\delta}_i + \frac{2\dot{a}}{a}\dot{\delta}_i + \left[\frac{c_{si}^2 k^2}{a^2}\delta_i - 4\pi G \rho_{tot} \sum_j \epsilon_j \delta_j\right] = 0$$

$\epsilon_j \equiv \frac{\rho_j}{\rho_{tot}}$ fraction of mass in species j

EXAMPLE: TWO COMPONENT FLUID

BARYONS AND PHOTONS

IN RADIATION DOMINATED ERA

$$\Rightarrow \frac{\dot{a}}{a} = \frac{1}{2t}$$

FOR $k \ll k_J$ AND ASSUME SMOOTH PHOTONS ($\delta_\gamma = 0$)

$\epsilon_b \ll 1$:

$$\ddot{\delta}_b + \frac{\dot{\delta}_b}{t} = 0$$

$$\Rightarrow \delta_b(t) = \delta_b(t_i)\left[1 + A \ln\left(\frac{t}{t_i}\right)\right]$$

VERY SLOW GROWTH

CURVATURE DOMINATION:

$$4\pi G \rho_{tot} = \frac{3}{2}\frac{\rho_{tot}}{\rho_{crit}} \ll t^{-2}$$

$$\Rightarrow \ddot{\delta}_b + \frac{2}{t}\dot{\delta}_b + \frac{3}{2t^2}\frac{\rho_{tot}(t)}{\rho_{crit}(t)}\delta_b \simeq \ddot{\delta}_b + \frac{2}{t}\dot{\delta}_b = 0$$

$$\Rightarrow \delta_b(t) = A + Bt^{-1}$$

$\Rightarrow$ SMALL PERTURBATIONS DO NOT GROW !

$\Rightarrow$ IN GENERAL DURING RD PERTURBATIONS IN NON-RELAT. COMPONENTS DO NOT GROW

- IN RD OR CURVAT. DOMINATION UNIVERSE EXPANDS TOO FAST FOR MATTER PERTURB. TO GROW !

PHOTONS: $\rho_\gamma = \frac{\pi^2}{15} T^4$

(Stefan-Boltzmann law)

BEFORE ELECTRONS AND PROTONS COMBINE TO FORM NEUTRAL HYDROGEN

$\Rightarrow$ BARYONS AND PHOTONS ARE TIGHTLY COUPLED

PRESSURE FROM PHOTONS:

$$c_s^2 = \frac{1}{3}$$

$$\Rightarrow k_{J\text{-PHYS}} \equiv \frac{k_J}{a} = \left(\frac{4\pi G \rho_{tot}}{c_s^2}\right)^{1/2}$$

$$= \left(\frac{4\pi^3}{5}\right)^{1/2} \frac{T^2}{m_{PL}}$$

$\Rightarrow$ physical Jeans Mass in baryons:

$$M_{B\text{-}J} = \frac{4\pi}{3} \rho_B \left(\frac{\pi}{k_{J\text{-PHYS}}}\right) \simeq 5.4 \times 10^{18} (\Omega_b h^2) T_{eV}^{-3} M_\odot$$

SIZE OF UNIVERSE (horizon): $d_H = \frac{c}{H} \sim t$

$\Rightarrow$ BARYONIC MASS INSIDE HORIZON

$$M_{B\text{-HOR}} \equiv \frac{4\pi}{3} \rho_B d_H^3 = \frac{t^3}{(\pi/k_{J\text{-PHYS}})^3} M_{J\text{-}B}$$

IN RD FROM 1st FRIEDMANN EQN

$$t^{-1} = \left(\frac{32\pi G \rho_{tot}}{3}\right)^{1/2}$$

$$\Rightarrow \frac{\pi}{k_{J\text{-PHYS}}} = \left(\frac{8}{3}\right)^{3/2} \pi c_s t$$

$$\Rightarrow \frac{M_{B-J}}{M_{B-HOR}} = \left(\frac{8}{3}\right)^{3/2} (\pi c_s)^3 \simeq 26$$

$\Rightarrow$ DURING RD BARYON JEANS MASS IS LARGER THAN BARYON MASS INSIDE HORIZON.

IF THERE ARE UNSTABLE MODES (?) THEY CAN NOT BE TREATED IN NEWTONIAN ANALYSIS

AFTER RECOMBINATION ($e^- + p \rightarrow H$)

PRESSURE ONLY BY HYDROGEN

$$c_s^2 = \frac{5}{3} \frac{T_B}{m}$$

AFTER DECOUPLING TEMPERATURE OF BARYONS SCALES AS a^{-2} (PHOTONS: a^{-1})

$$T_B = \frac{a_{rec}}{a} T_\gamma = \frac{T_\gamma^2}{T_{rec}}$$

$$\Rightarrow M_{B\rightarrow} \simeq 1.3 \times 10^5 (\Omega_{B,0} h^2)^{-1/2} \left(\frac{z}{1100}\right)^{3/2} M_\odot$$

($M_\odot \approx 2 \cdot 10^{30}$ kg)

$\Rightarrow$ AFTER RECOMBINATION JEANS MASS IN BARYONS MUCH SMALLER, $\Rightarrow$ ONLY SMALL, SUB-HORIZON OVERDENSITIES CAN GROW